MEMOIR

On the equations of relative motion of systems of bodies;

By G. CORIOLIS.

In a Memoir in the 21st issue of the *Journal of the Polytechnic School (Journal de l'École Polytechnique)*, I showed that in order to apply the principle of living forces[1] to the relative motions of systems driven with coordinated planes having any motion in space, it was enough to add to the given forces other forces opposite to those that are capable of forcing the material points to remain invariably bound to the moving planes to which the relative motions are related.

I pointed out in this Memoir that the proposition which is its object cannot be applied in general to other equations of motion than those of the living forces; but I had not then examined whether there are circumstances in which the step it provides can be applied to certain equations of motion; and whether, in the sense in which it does not apply, a simple expression of the new terms of correction can be given.

This is the question I have dealt with in the Memoir I am presenting today. I give this general proposition, namely: that to establish any equation of relative motion of any system of bodies or of any machine, it is enough to add to the existing forces two species of supplementary forces. The first ones are always those which must be taken into account for the equation of the living forces, that is to say, they are forces opposed to those which are capable of maintaining the material points invariably linked to the moving planes. The second ones are directed perpendicularly to the relative velocities and to the axis of rotation of the moving planes; they are equal to the twice the product of the angular velocity of the moving planes multiplied by the quantity of relative motion projected on a plane perpendicular to this axis.

These latter forces have the greatest analogy with ordinary centrifugal forces.

To highlight this analogy, it is enough to note that the centrifugal force is equal to the momentum multiplied by the angular velocity of the tangent to the curve described, and that it is directed perpendicularly to the velocity and in the osculating plane, that is to say, also perpendicularly to the axis of rotation of the tangent. Thus, in order to pass from these ordinary centrifugal forces to the second forces whose doubles enter into the preceding statement, one has only to replace the angular velocity of the tangent by that of the moving planes, and to substitute for the direction of the axis of rotation of this tangent, the direction of the axis of rotation of these same moving

[1] Understanding Coriolis' use of the term "living forces" requires placing the use of the term in the context of the time that Coriolis was writing. In early classical mechanics, the term 'force vive' (French for 'living force') was used to describe the quantity of motion associated with an object's velocity. Over time, the understanding of "force vive" matured into the modern concept of kinetic energy.

planes. In other words, it suffices to substitute for everything that relates in magnitude and direction to the rotation of the tangent, that which relates to the moving planes, and to double the forces thus obtained.

It is because of this analogy that I thought I should give these new forces the name of *compound centrifugal forces*: they participate in relative motion by the quantity of motion, and in the motion of moving planes by the use of their axis of rotation and their angular velocity.

We will therefore say that in order to pose an equation of relative motion, which is not that of the living forces, it is necessary to introduce, in addition to this equation, the doubles of *the compound centrifugal forces*.

The directions of these second additional forces being perpendicular to the relative velocities, one sees immediately that they disappear in the equation of the living forces for relative motion, since one uses in the latter only the components of the forces in the direction of the relative velocities.

It is in this disappearance of these *compound centrifugal forces* that consists the theorem that I presented to the Academy of Sciences in 1831. It now becomes a special case of the more general statement on the introduction of these *compound centrifugal forces.*

There are in certain circumstances other equations where these compound centrifugal forces still disappear; these are those that relate to relative motions which operate in moving planes that can remain parallel to the axis of rotation of these planes. It is clear indeed that the centrifugal forces perpendicular to this axis of rotation, as well as to the relative velocities, disappear when we are only concerned with the projections of the relative forces onto the planes in which the motions take place.

We can also present the introduction of the *compound centrifugal forces* by using in the statements the relative virtual velocities that were used to obtain each equation of motion. We thus arrive at this proposition that the two kinds of additional terms which enter into an equation of relative motion are, 1) the virtual moments of the same forces that enter into the equation of living forces, and 2) the doubles of the sums of the areas of the parallelograms constructed on the relative velocities and the virtual velocities, these areas being projected on a plane perpendicular to the axis of rotation of the moving planes.

This last statement shows in which cases these second additional terms disappear, no longer in isolation, as seen only by the direction of the *compound centrifugal forces*, but as a whole.

Thus, if the axis of rotation has a fixed position in space, and consequently fixed also with respect to the moving planes, the compound centrifugal forces always disappear in the equation of motion of the projection of the center of gravity on a line parallel to this axis.

If the center of gravity can only move on a fixed line with respect to the moving plane, the compound centrifugal forces disappear in the equation of relative motion of this center.

In the equations of areas, the compound centrifugal forces disappear only in the very special case where the axis of rotation of the moving planes is fixed in direction, and the areas are taken on a plane that is perpendicular to it and around an axis with respect to which the moments of inertia of the system do not change during the relative motion.

Here are the demonstrations of the propositions just stated.

Let us designate by *x, y, z,* coordinates related to the moving planes; by L=o, etc., the equations of connections of the moving points, which are assumed to be expressed by these relative coordinates *x, y, z*. Let us represent by λ, etc., available coefficients; by X_{e5}, Y_{e5}, Z_{e5}, the forces that would produce the motions due to the connection with the moving planes; and finally, by *abc*, *a'b'c'*, *a"b"c"*, the cosines of the angles that the moving axes make with fixed axes.

It has been established, in the memoir already cited, Page 275 of the Journal of the Polytechnic School, XXI[st] issue, that for one of the material points of which *m* is the mass, one has

$$
\begin{aligned}
m\frac{d^2x}{dt^2} &+ 2(adb + a'db' + a''db'')\,mdy \\
&+ 2(adc + a'dc' + a''dc'')\,mdz = X - X_e + \lambda\frac{dL}{dx} + \text{etc.}, \\
m\frac{d^2y}{dt^2} &+ 2\left(\frac{bda}{dt} + \frac{b'da'}{dt} + \frac{b''da''}{dt}\right)m\frac{dx}{dt} \\
&+ 2\left(\frac{bdc}{dt} + \frac{b'dc'}{dt} + \frac{b''dc''}{dt}\right)m\frac{dz}{dt} = Y - Y_e + \lambda\frac{dL}{dy} + \text{etc.}, \\
m\frac{d^2z}{dt^2} &+ 2(cda + c'da' + c''da'')\,mdx \\
&+ 2(cdb + c'db' + c''db'')\,mdy = Z - Z_e + \lambda\frac{dL}{dz} + \text{etc.}
\end{aligned}
$$

Let us designate as usual by *p,q,r,* the three projections of the angular speed of rotation of the moving planes on these same planes, or in other words, the three angular speeds of these planes taken around their axes. These equations will thus become

$$
\text{(A)}\quad
\begin{aligned}
m\frac{d^2x}{dt^2} &= 2\left(rm\frac{dy}{dt} - qm\frac{dz}{dt}\right) + X - X_e + \lambda\frac{dL}{dx} + \text{etc.}, \\
m\frac{d^2y}{dt^2} &= 2\left(pm\frac{dz}{dt} - rm\frac{dx}{dt}\right) + Y - Y_e + \lambda\frac{dL}{dy} + \text{etc.}, \\
m\frac{d^2z}{dt^2} &= 2\left(qm\frac{dx}{dt} - pm\frac{dy}{dt}\right) + Z - Z_e + \lambda\frac{dL}{dz} + \text{etc.}
\end{aligned}
$$

We see here, in the expression of the forces that must be considered in relative motions, two additional terms; some are expressed by $-X_e$, $-Y_e$, $-Z_e$, and are forces opposed to those that would be capable of forcing the moving points to remain invariably linked to the moving coordinate planes; the others are expressed by

$$
\begin{aligned}
&2\left(rm\frac{dy}{dt} - qm\frac{dz}{dt}\right), \\
&2\left(pm\frac{dz}{dt} - rm\frac{dx}{dt}\right), \\
&2\left(qm\frac{dx}{dt} - pm\frac{dy}{dt}\right).
\end{aligned}
$$

Note that if x, y, z, x', y', z', are the projections of two lengths r and r'; the parallelogram constructed on r and r', and whose expression is $rr' \sin (rr')$, has projections on the coordinate planes

$$
\begin{gathered}
(xz' - zx'),\\
(xy' - yz'),\\
(yx' - xy').
\end{gathered}
$$

The above expressions can also be the projections onto the coordinate axes of a length equal to $rr' \sin (rr')$, which will be carried perpendicular to the plane of the two lines r and r' and will be located on the same side, with respect to the direction going from r to r', as the z axis is with respect to the direction going from y to x.

According to this remark, the above expressions in $p, q, r, dx, dy, dz,$ will be the doubles of the components along the axes of a force directed perpendicular to the plane of the axis of rotation and of the relative velocity, which force will have as its magnitude the product of the angular velocity $\sqrt{p^2 + q^2 + r^2}$, multiplied by the projection or component, in a plane perpendicular to the axis of rotation, of the quantity of motion due to the relative velocity of the material point. The direction in which this force must be carried, with respect to a motion moving from the axis of rotation to the relative velocity, will be the same as that of the axis of rotation with respect to the velocity of rotation.

The introduction of the above terms therefore amounts to that of a new force, which has a complete analogy with the ordinary centrifugal force.

In fact, if we denote by ω the angular velocity with which turns the tangent to the curve described by a material point whose mass is m, by v the velocity, the ordinary centrifugal force can be put in the form

$$\omega m v,$$

that is, it is the product of this angular velocity multiplied by the quantity of motion of the material point; moreover, its direction is both perpendicular to the velocity v and to the axis of rotation of the tangent, since it is in the osculating plane which is the one in which the tangent rotates.

We see therefore that, in order to pass from the ordinary centrifugal forces to the second forces whose doubles enter into the equations of relative motion, it is sufficient to substitute at the same time to the axis of rotation of the tangent, to the angular velocity, and to the quantity of motion of the moving point, the axis of rotation of the moving planes, the angular velocity of these planes, and the quantity of motion projected on a plane perpendicular to this axis.

These second centrifugal forces, resulting from the simultaneous use of the relative motions and the motions of the moving planes, can be called *compound centrifugal forces*. We thus arrive at this proposition, *that the expressions of the forces to be added to the given forces in order to have the expressions of the forces in the relative motions are, 1) those which are opposed to the* *forces capable of producing on each point the motion it would have if it were linked to the moving planes, 2) the doubles of the compound centrifugal forces.*

We see immediately that these second forces disappear in the equation of the dynamic forces like ordinary centrifugal forces, since they are directed perpendicular to the relative velocities, and that the equation of the dynamic forces is obtained only by projecting the relative forces onto the direction of the relative velocities themselves.

They also disappear when the relative motion must take place in planes parallel to the axis of rotation of the moving planes, since the equations of motion in these planes will not include forces which, being perpendicular to the axis of rotation, will also be perpendicular to the planes in which the motions take place.

We can see that these second forces disappear in the equation of the living forces like the ordinary centrifugal forces, since they are directed perpendicularly to the relative velocities, and that we only obtain the equation of the living forces by projecting the relative forces on the direction of the relative velocities themselves.

They also disappear when the relative motions must be made in planes parallel to the axis of rotation of the moving planes, since the equations of motion in these planes will not contain forces which, being perpendicular to the axis of rotation, will also be perpendicular to the planes in which the motions take place.

We can still, if you wish, give another statement of the correction terms due to these compound centrifugal forces, when we consider them no longer isolated in the expression of each force, but in any equation of motion obtained by choosing a system of relative virtual velocities.

By calling $\delta x, \delta y, \delta z$, etc., the components of the virtual velocities taken in the relative motion, that is to say velocities compatible with the relative bonds expressed by L=o, etc., one will have, by indicating by Σ a sum extending to all the points which enter the equation L=o,

$$\Sigma\left(\frac{dL}{dx}\delta x + \frac{dL}{dy}\delta y + \frac{dL}{dz}\delta z\right) = 0.$$

Multiplying each of the equations (A) by the corresponding virtual velocity and adding them all together, the terms $\frac{dL}{dx}, \frac{dL}{dy}, \frac{dL}{dz}$, etc. will go away, that is to say, those forces that come from the bonds will be eliminated, and it will become

$$\text{(B)}\quad \begin{aligned} \Sigma m\left(\frac{d^2x}{dt^2}\delta x + \frac{d^2y}{dt^2}\delta y + \frac{d^2z}{dt^2}\delta z\right) &+ 2p\Sigma m\left(\frac{dy\delta z - dz\delta y}{dt}\right) \\ &+ 2q\Sigma m\left(\frac{dz\delta x - dx\delta z}{dt}\right) \\ &+ 2r\Sigma m\left(\frac{dx\delta y - dy\delta x}{dt}\right) \\ = \Sigma(X\delta x + Y\delta y + Z\delta z) &- \Sigma(X_e\delta x + Y_e\delta y + Z_e\delta z). \end{aligned}$$

This is the general formula that will give all equations relating to relative motions.

These equations, instead of having two terms as for absolute motions, always contain four types of terms that depend on 1) second differentials, 2) first differentials, 3) the variables themselves, 4) terms that depend on the given forces, which, depending on the cases, will depend on the coordinates or on their first differentials.

We see that there are two types of additional terms. Some are due to the forces X_e, Y_e, Z_e, that thus depend on the coordinates *x, y, z,* which are the unknowns of the problem. The others depend on the differentials *dx, dy, dz* of these unknowns.

Note that the factor

$$m(dy\delta z - dz\delta y)$$

is nothing other than the area of the parallelogram constructed on the projection of the effective speed and the virtual speed on the *yz* plane; thus

$$\Sigma m(dy\delta z - dv\delta y)$$

will be the algebraic sum of all similar areas for all points in the system.

Each of these areas in space can be expressed by........$mds\delta s$ sin $(\widehat{ds\delta s})$. Representing by λ, μ, *v*, the angles that the perpendicular to its plane makes with the coordinate axes, we will have

$$\Sigma m\left(\frac{dy}{dt}\delta z - \frac{dz}{dt}\delta y\right) = \Sigma m\frac{ds}{dt}\delta s \sin(\widehat{ds\delta s})\cos\lambda,$$
$$\Sigma m\left(\frac{dz}{dt}\delta x - \frac{dx}{dt}\delta z\right) = \Sigma m\frac{ds}{dt}\delta s \sin(\widehat{ds\delta s})\cos\mu,$$
$$\Sigma m\left(\frac{dx}{dt}\delta y - \frac{dy}{dt}\delta x\right) = \Sigma m\frac{ds}{dt}\delta s \sin(\widehat{ds\delta s})\cos\nu.$$

Representing by *α, β,γ,* the angles that the instantaneous axis of rotation of the moving planes makes with these axes, and by *ω* their angular speed of rotation around this axis; we will have

$$p = \omega\cos\alpha,$$
$$q = \omega\cos\beta,$$
$$r = \omega\cos\gamma.$$

The sum of the terms in question in equation (B) thus becomes equal to

$$2\omega\Sigma m\frac{ds}{dt}\delta s\sin(\widehat{ds\delta s})(\cos\alpha\cos\lambda + \cos\beta\cos\mu + \cos\gamma\cos\nu).$$

We see that it is the sum of the projections of the areas $m \frac{ds}{dt} \delta s \sin(\widehat{ds \delta s})$ on a plane perpendicular to the axis of rotation. Thus we can say that, *in order to have an equation of relative motion, it is necessary to add to the terms ordinarily existing for absolute motion, first that which comes from the forces that are capable of forcing the points to remain invariably related to the moving planes, and furthermore, a term that is equal to twice the angular velocity of rotation of the moving axes multiplied by the sum of the projections on a plane perpendicular to the axis of rotation of these planes, of all the areas of the parallelograms included between the effective quantities of motion and the virtual velocities.*

In the case of the equation of living forces, each area is zero, since the virtual one coincides with the effective velocity; the sum of these areas is therefore also zero, and the last term of corrections disappears. This remark precisely forms the theorem I gave on the principle of live forces in relative motion.

There is another fairly general case where these areas also disappear, namely where the relative and virtual motions occur for each point in a plane parallel to the axis of rotation of the relative planes. It is clear, in fact, that the areas between these two speeds become zero when projected onto a plane perpendicular to the axis of rotation. Thus, in this case, all the equations of motion, for example, those of the center of gravity and the areas, are the same for relative motion by adding only the forces $-X_e$, $-Y_e$, $-Z_e$.

When we want to have equations that, for this relative motion, refer to the center of gravity, that is, which result from virtual velocities equal and parallel to one of the moving axes, it is sufficient to add together all the equations (A) that refer to the same coordinate.

Representing by ξ, *n*, and ζ, the coordinates of the center of gravity, relative to the moving axes, and setting Σm = M, we have

$$\Sigma m \frac{d^2x}{dt^2} = \mathrm{M} \frac{d^2\zeta}{dt^2}$$

and

$$\Sigma m \frac{dx}{dt} = \mathrm{M} \frac{d^2\zeta}{dt};$$

which gives for the sums in question, where the forces $\lambda \frac{dZ}{dx}$, etc., always disappear,

$$\text{(C)} \quad \begin{aligned} \mathrm{M}\frac{d^2\xi}{dt^2} + 2\mathrm{M}\left(q\frac{d\zeta}{dt} - r\frac{d\eta}{dt}\right) &= \Sigma \mathrm{X} - \Sigma \mathrm{X}_e, \\ \mathrm{M}\frac{d^2\eta}{dt^2} + 2\mathrm{M}\left(r\frac{d\xi}{dt} - p\frac{d\zeta}{dt}\right) &= \Sigma \mathrm{Y} - \Sigma \mathrm{Y}_e, \\ \mathrm{M}\frac{d^2\zeta}{dt^2} + 2\mathrm{M}\left(p\frac{d\eta}{dt} - q\frac{d\xi}{dt}\right) &= \Sigma \mathrm{Z} - \Sigma \mathrm{Z}_e. \end{aligned}$$

If, in the relative motion, the center of gravity of the system stays on a straight line parallel to the axis of rotation of the moving planes, we have

$$\frac{d\xi}{p} = \frac{d\eta}{q} = \frac{d\zeta}{r},$$

or else

$$q\frac{d\zeta}{dt} - r\frac{d\eta}{dt} = 0,$$
$$r\frac{d\xi}{dt} - p\frac{d\zeta}{dt} = 0,$$
$$p\frac{d\eta}{dt} - q\frac{d\xi}{dt} = 0.$$

Thus, the equations above have no second term, and are reduced to

$$M\frac{d^2\xi}{dt^2} = \Sigma X - \Sigma X_e,$$
$$M\frac{d^2\eta}{dt^2} = \Sigma Y - \Sigma Y_e,$$
$$M\frac{d^2\zeta}{dt^2} = \Sigma Z - \Sigma Z_e.$$

Only one of these three can remain the same way without the second term of correction, if the management of the relative coordinate, which between the second order differential, is perpendicular both to the axis of rotation and to the speed of the center of gravity. Because then, if it's the coordinate ξ, for example, we have

$$\frac{d\eta}{q} = \frac{d\zeta}{r}.$$

So, when only two axes of coordinates are mobile, and the third, that of ξ, for example, remains parallel to the axis of rotation to which the quantities *p, q, r,* as we then have *q = o, r = o*, we will have the equation

$$M\frac{d^2\xi}{dt^2} = \Sigma X - \Sigma X_e.$$

Finally, if the center of gravity moves on a given line compared to the mobile plane, taking it for the axis of ζ, we will have $\frac{d\eta}{dt} = 0, \frac{d\xi}{dt} = 0,$ and consequently,

$$M\frac{d^2\xi}{dt^2} = \Sigma X - \Sigma X_e.$$

If now we want to examine what the equations of the areas become, it will be necessary, in equation (B), to take the virtual rotation speeds around one of the axes of the coordinates, for example, to ask

$$\delta z = 0, \quad x\delta x + y\delta y = 0,$$

or

$$\delta z = 0, \quad \frac{\delta x}{y} = \frac{\delta y}{-x}.$$

We will thus obtain

$$\Sigma m\left(y\frac{d^2x}{dt^2} - x\frac{d^2y}{dt^2}\right) - 2r\Sigma m\,(xdx + ydy)$$
$$+ \Sigma m\,(px + qy)\,dz = \Sigma(Xy - Xx) - \Sigma(X_e y - Y_e x).$$

This equation, as well as the other two similar ones we would obtain for virtual rotation speeds around the other coordinate axes, do not simplify in general.

If the axis of rotation of the mobile planes has a constant direction in the space, we know that it is thus compared to the mobile axes, we can take it for the *z* axis, and we will have

$$p = 0, \quad q = 0.$$

The above equation then becomes

$$\Sigma m\left(\frac{yd^2x - xd^2y}{dt^2}\right) - 2r\Sigma m\,(xdx + ydy)$$
$$= \Sigma(Xy - Yx) - \Sigma(X_e y - Y_e x).$$

Thus this relation is always among the equations of relative motion when the axis of rotation of the motion that drives the moving planes has a constant direction in space. If the moving points, in their relative motion, do not change their distance from the axis around which the areas are taken, that is, from which the *x* and *y* coordinates are counted here, the term of correction $2r\Sigma m(xdx + ydy)$ disappears in the above equation.

If the XY forces are directed towards the origin of the coordinates, they disappear from this equation. The same will be true for the forces X_e, Y_e, if the axis of rotation keeps a constant direction that is taken as the *z* axis, and if the angular velocity of rotation of the moving planes is uniform, in other words, if *r* is constant and equal to *ω,* we will have therefore

$$\Sigma m\left(\frac{yd^2x - xd^2y}{dt^2}\right) = 2\omega\Sigma m(xdx + ydy).$$

Designating by A the moment of variable inertia of the system at any instant, and by λ the sum of the areas described on the *x, y* plane, we will have, integrating between two instants, and indicating the first limit by the index zero,

$$\frac{d\lambda}{dt} - \frac{d\lambda_0}{dt} = 2\omega(A - A_0).$$

Thus, under the previous hypotheses, the differentials of the areas in the relative motions, when projected onto a plane perpendicular to the axis of rotation, grow like the moments of inertia.